Vicious walkers and hook Young tableaux

2003 J. Phys. A: Math. Gen. 36 3033

(http://iopscience.iop.org/0305-4470/36/12/311)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 3033–3048 PII: S0305-4470(03)54082-3

Vicious walkers and hook Young tableaux

Kazuhiro Hikami and Takashi Imamura

Department of Physics, Graduate School of Science, University of Tokyo, Hongo 7-3-1, Bunkyo, Tokyo 113-0033, Japan

E-mail: hikami@phys.s.u-tokyo.ac.jp and imamura@monet.phys.s.u-tokyo.ac.jp

Received 27 September 2002, in final form 28 November 2002
Published 12 March 2003
Online at stacks.iop.org/JPhysA/36/3033

Abstract
We consider a generalization of the vicious walker model. Using a bijection map between the path configuration of the non-intersecting random walkers and the hook Young diagram, we compute the probability concerning the number of movements of the walker. Applying the saddle point method, we reveal that the scaling limit gives the Tracy–Widom distribution, which is the same with the limit distribution of the largest eigenvalues of the Gaussian unitary ensemble.

PACS numbers: 05.40.Fb, 05.50.+q

1. Introduction

Since it has been shown that the path configuration of the random vicious walkers [1] is related to the Young tableaux [2–4], much attention has been paid to the statistical combinatorial problems, which are intimately related with the Young tableaux. Among these are the random permutation [5], the random word [6], the point process [7, 8], the random growth model (the polynuclear growth model, oriented digital boiling model) [9, 10], the queuing theory [11], and so on. It is interesting that the scaling limits of these models have the universality that the fluctuation is of order $N^{1/3}$ with the mean being of order N. It is also of interest that the asymptotic distribution of appropriately scaled variables is described by the Tracy–Widom distribution, which was originally identified with the limit distribution for the largest eigenvalue of the Gaussian unitary random matrix [12]; see [13–16] for a review.

In this paper, motivated by results in [17] and conjectures in [18], we introduce a physical model of the vicious walkers based on the hook Young tableaux. We study the scaling limit of certain probability, and we clarify a relationship with the Tracy–Widom distribution.

For the convention we use later, we define the (M, N)-hook Schur functions, which are sometimes called the supersymmetric Schur functions [19] (see also [20–22]), and briefly we denote some properties of the hook Young tableaux. We set $\mathbf{B} = \mathbf{B}_+ \sqcup \mathbf{B}_-$, and

$$\mathbf{B}_+ = \{\epsilon_1, \ldots, \epsilon_M\} \qquad \mathbf{B}_- = \{\epsilon_{M+1}, \ldots, \epsilon_{M+N}\}. \tag{1.1}$$

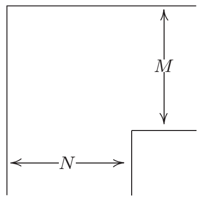


Figure 1. The (M, N)-hook Young diagram must be contained in the above 'hook' region.

Hereafter we call i the positive (negative) symbol when $\epsilon_i \in \mathbf{B}_+$ ($\epsilon_i \in \mathbf{B}_-$). We fix an ordering in $\mathbf{B}$ as

$$\epsilon_1 \prec \epsilon_2 \prec \cdots \prec \epsilon_{M+N}. \tag{1.2}$$

It should be noted that, although we use an ordering (1.2), the following discussion can be applied for any other choices of ordering with $|\mathbf{B}_+| = M$ and $|\mathbf{B}_-| = N$. For a given Young diagram λ, the semi-standard Young tableaux (SSYT) T is given by filling a number $1, 2, \ldots, M + N$ in λ by the following rules:

- the entries in each row are increasing, allowing the repetition of positive symbols, but not permitting the repetition of negative symbols;
- the entries in each column are increasing, allowing the repetition of negative symbols, but not permitting the repetition of positive symbols.

We define the weight for the SSYT T as

$$\mathrm{wt}(T) = \sum_{a=1}^{M+N} m_a \epsilon_a \tag{1.3}$$

where m_a is the number of a in T. Then the hook Schur function $S_\lambda(x, y)$ is given by

$$S_\lambda(x, y) = \sum_{\text{SSYT } T \text{ of shape } \lambda} \mathrm{e}^{\mathrm{wt}(T)}. \tag{1.4}$$

Here we have used

$$\begin{cases} x_i = \mathrm{e}^{\epsilon_i} & \text{for} \quad \epsilon_i \in \mathbf{B}_+ \\ y_j = \mathrm{e}^{\epsilon_{M+j}} & \text{for} \quad \epsilon_{M+j} \in \mathbf{B}_-. \end{cases}$$

The Schur function $s_\lambda(x)$ in usual sense corresponds to a case of $\mathbf{B}_- = \emptyset$ ($N = 0$), and the hook Schur function S_λ with $\mathbf{B}_+ = \emptyset$ ($M = 0$) reduces to the Schur function for the conjugate partition λ':

$$S_\lambda(x, 0) = s_\lambda(x) \qquad S_\lambda(0, y) = s_{\lambda'}(y). \tag{1.5}$$

Due to the rule of filling a number, the Young diagram λ should be contained in the (M, N)-hook (see figure 1), and we have

$$S_\lambda(x, y) = \sum_{\mu \subset \lambda} s_\mu(x) s_{\lambda'/\mu'}(y).$$

Furthermore, when λ contains the partition (N^M), we have

$$S_\lambda(x, y) = s_\mu(x) s_\nu(y) \prod_{i=1}^{M} \prod_{j=1}^{N} (x_i + y_j)$$

where the partitions μ and ν are defined from λ by $\mu_i = \lambda_i - N$ and $\nu_j = \lambda'_j - M$, respectively.

The Jacobi–Trudi formula helps us to write the hook Schur function as

$$S_\lambda(x, y) = \det(c_{\lambda_i+j-i})_{1\leqslant i,j\leqslant \ell(\lambda)}. \tag{1.6}$$

Here $\ell(\lambda)$ is the length of λ, and c_n is given by the generating function,

$$H(t; x)E(t; y) = \sum_{n=0}^{\infty} c_n t^n \tag{1.7}$$

with

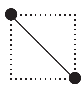
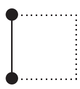
$$H(t; x) = \prod_j \frac{1}{1 - t x_j} \qquad E(t; y) = \prod_j (1 + t y_j). \tag{1.8}$$

This paper is organized as follows. In section 2 we introduce a model of vicious walkers as a generalization of the original model [1]. As far as we know, this model is presented for the first time in this paper. We define the bijection from path configurations of the vicious walker to the hook Young diagram. Especially, we show a relationship between the length of the Young diagram and the number of movements of the first walker. This type of bijection was first given in [2, 3] for the original vicious walker model. In section 3 we give the probability of $\ell(\lambda) \leqslant \ell$ in terms of the Toeplitz determinant. We further study the scaling limit of this probability based on the transformation identity from the Toeplitz determinant to the Fredholm determinant [23–25] in section 4. We apply the saddle point method to the Fredholm determinant following [10, 26], and we show that the scaling limit coincides with the Tracy–Widom distribution for the Gaussian unitary ensemble (GUE) [12]. In section 5 we consider some simple examples as a reduction of our model. Both the Meixner and the Krawtchouk ensembles can be regarded as a reduction of our vicious walker model. The final section contains our conclusion and discussion. We briefly comment on the random word related to the hook Young tableaux.

2. Vicious walker

We define a model of the random walkers, which is related to the hook Schur function (1.4). The model is a generalization of that introduced in [1] and, as clarified later, an algebraic property of the partition function is nothing but an identity in [17].

The evolution rule of vicious walkers is defined as follows. Initially there are infinitely many walkers at $\{\dots, -2, -1\}$, and we call each walker P_j whose initial point is $-j$. A walker is movable rightward if its right site is vacant. Walkers $P_{j+1}, P_{j+2}, \dots$ are called successors of a walker P_j if they are next to each other in the order of the indices. We consider two types of time evolution (we assume that there are totally $M + N$ time steps); the first M-steps are referred to as 'normal' time evolution, and the following N-steps are 'super' time evolution. At a 'normal' time evolution, a movable walker either stays or moves to its right together with an arbitrary number of its successors. Thus we draw

move :  stay : 

On the other hand, at a 'super' time evolution, a walker can move to its right any number of lattice units, although P_j cannot over-pass a position of P_{j-1} at previous time. To realize this

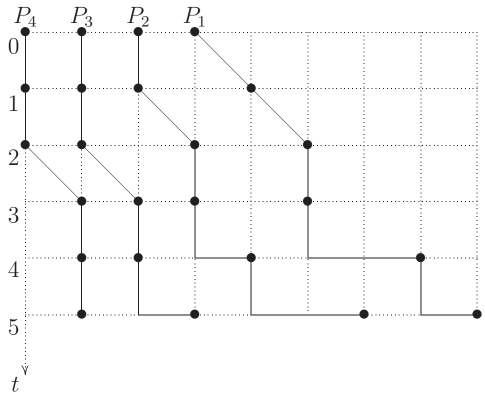
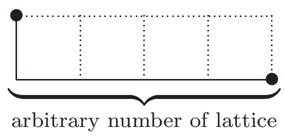


Figure 2. Example of path configuration. For $t \leqslant 3$, a process is a 'normal' evolution, while for $t \geqslant 4$ it becomes a super time evolution.

rule and to draw a non-intersecting path, it is convenient to depict this step as follows:



Each path of the vicious walkers is required not to intersect each other. We see that the original model of vicious walkers [1] corresponds to a case of $N = 0$. After $M + N$ time steps, we denote $L_j(n)$ as the number of right moves made by the walker P_j. Here, n is the total number of movements of walkers. In figure 2 we give an example of path configuration of vicious walkers. In this case, we consider in total five time steps ($M = 3$ and $N = 2$), and the total step of right movements is $n = 12$ with $(L_1, L_2, L_3, L_4) = (5, 4, 2, 1)$.

It is now well known for the model of the original vicious walkers [1] that we have the bijection from the path configuration of vicious walkers to the Young diagram [2]. This bijection can be easily generalized to our model as follows. For a path configuration (see, for example, figure 2), we draw Young tableaux $\lambda \vdash n$ with $\lambda'_j = L_j(n)$. We insert in the jth column from the top the times at which the jth particle made a movement to its right. Notice that, for a super time evolution, we prepare the number of times to be as many lattice units as the walker has moved. For instance, in the case of figure 2, we put '1 2 4 4 5' in the first column, as P_1 moves two lattice units to the right at time 4. Thus, the top row is the list of times at which the walkers made their first movement, the second row is the list of times at which the walkers made their second movement, and so on. It is clear that the normal time corresponds to the positive symbol $\mathbf{B}_+$ while the super time denotes the negative symbol $\mathbf{B}_-$ in the SSYT. The evolution rule supports a consistency with ordering (1.2) in $\mathbf{B}$, and we know that the map is indeed the bijection. Following this mapping, the path configuration in figure 2 is mapped to SSYT given in figure 3.

To summarize, we have a one-to-one correspondence between the path configuration and the SSYT; when n is the total number of moves of the vicious walkers, we have $\lambda \vdash n$, and the number $L_j(n)$ of right movements made by the jth walker is equal to the number of boxes in the jth column of the SSYT. Especially, the length $\ell(\lambda)$ of the partition coincides with $L_1(n)$.

1	2	3	3
2	4	5	
4	5		
4	5		
5			

Figure 3. Semi-standard (hook) Young tableaux with entries from $\mathbf{B}_+ = \{1, 2, 3\}$ and $\mathbf{B}_- = \{4, 5\}$.

3. Partition function and Toeplitz determinant

In the following, we consider a model where, after a total of n step movements of the right movers, every walker returns to its initial position by a total of n step left movements [3]. Here the number of normal (super) time evolution is supposed to be M_1 (N_1) in the first right moves, while the number of normal (super) time evolution is M_2 (N_2) in the next left moves returning to their initial positions. The definition of normal and super time evolution in the left movers simply follows from that of right movers as a mirror image. Applying the bijection in the previous section, the path configuration is denoted by pairs of SSYT $\lambda \vdash n$: one is (M_1, N_1)-hook Young tableaux and the other is (M_2, N_2)-hook tableaux.

We denote $d_\lambda(M, N)$ as the number of SSYT of shape λ with entries from $\mathbf{B}_+ \sqcup \mathbf{B}_-$ (with $|\mathbf{B}_+| = M$ and $|\mathbf{B}_-| = N$). By definition, we have $S_\lambda(\underbrace{t, \ldots, t}_{M}, \underbrace{t, \ldots, t}_{N}) = d_\lambda(M, N)t^n$ for $\lambda \vdash n$, and once the Young diagram λ is fixed the number of SSYT $d_\lambda(M, N)$ corresponds to the number of path configuration with fixed endpoints of right moves.

We are interested in the probability that the number of right movements of the first walker P_1 is less than ℓ:

$$\mathrm{Prob}(L_1 \leqslant \ell). \tag{3.1}$$

Here the probability 'Prob' is defined as follows. We assign the weight t (we set $0 < t < 1$) for every right and left move, and we regard the weight of the total number of n step walks as t^n. Then each configuration of random walk, in which every walker returns to its initial position after total $2n$ step, is realized with a probability t^{2n}/Z. An explicit form of the normalization factor Z will be given later. Based on the bijection map studied in the previous section, we find that the probability (3.1) is given explicitly by

$$\mathrm{Prob}(L_1 \leqslant \ell) = \frac{1}{Z}\sum_n \left(\sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leqslant \ell}} d_\lambda(M_1, N_1)\, d_\lambda(M_2, N_2) \right) t^{2n}. \tag{3.2}$$

Note that a normalization factor is set to be $\lim_{\ell\to\infty} \mathrm{Prob}(L_1 \leqslant \ell) = 1$.

To relate this probability to the random matrix theory, we follow a method in [6]. Applying the Gessel formula to equation (1.6), we have

$$\begin{aligned} \sum_{\ell(\lambda)\leqslant\ell} S_\lambda(x, y) S_\lambda(z, w) &= \frac{1}{\ell!(2\pi)^\ell} \int_{-\pi}^{\pi} \mathrm{d}\theta \prod_{1\leqslant j<k\leqslant\ell} |\mathrm{e}^{\mathrm{i}\theta_j} - \mathrm{e}^{\mathrm{i}\theta_k}|^2 \\ &\quad \times \prod_{j=1}^{\ell} H(\mathrm{e}^{\mathrm{i}\theta_j}; x) E(\mathrm{e}^{\mathrm{i}\theta_j}; y) H(\mathrm{e}^{-\mathrm{i}\theta_j}; z) E(\mathrm{e}^{-\mathrm{i}\theta_j}; w) \\ &= D_\ell(\varphi) \end{aligned} \tag{3.3}$$

where $\varphi(z)$ is defined by

$$\varphi(z) = \prod_{i,j} \frac{1 + y_i z^{-1}}{1 - x_j z^{-1}} \frac{1 + w_i z}{1 - z_j z}. \tag{3.4}$$

We have used $D_\ell(\varphi)$ as the Toeplitz determinant for the function $\varphi(z)$. $D_\ell(\varphi)$ is the determinant of the $\ell \times \ell$ matrix where an (i, j) element is given by φ_{i-j} with $\varphi(z) = \sum_{n\in\mathbb{Z}} \varphi_n z^n$. We note that equation (3.3) was also given in [17]. Thus, our model of random walkers corresponds to a point process in [17], which was introduced as a generalization of [7]. We note that the strong Szegö limit theorem gives a generalization of the Cauchy formula:

$$\lim_{\ell\to\infty} D_\ell(\varphi) = \prod_{i,j,m,n} \frac{(1 + x_i w_n)(1 + y_j z_n)}{(1 - y_j w_n)(1 - x_i z_m)}. \tag{3.5}$$

We now apply a principal specialization ps which sets $x_i = aq^i$ and $y_j = bq^j$ [22]. In general, we have

$$\mathrm{ps}(S_\lambda(x, y)) = S_\lambda(\underbrace{aq, aq^2, \ldots}_{\infty}, \underbrace{bq, bq^2, \ldots}_{\infty}) = q^{\sum_{i=1}^{\ell(\lambda)} i\lambda_i} \prod_{(i,j)\in\lambda} \frac{a + bq^{j-i}}{1 - q^{\lambda_i - j + \lambda'_j - i + 1}}$$

and, for a case of $\lambda \vdash n$ and (M, N)-hook Young diagram, by definition, by setting $a = b = t$ and $q = 1$, we have

$$\mathrm{ps}_{a=b=t;q=1}(S_\lambda(x_1, \ldots, x_M, y_1, \ldots, y_N)) = d_\lambda(M, N)t^n.$$

As a result, from equation (3.3) we obtain the partition function as

$$\sum_n \sum_{\substack{\ell(\lambda)\leqslant\ell \\ \lambda\vdash n}} d_\lambda(M_1, N_1) d_\lambda(M_2, N_2) t^{2n} = D_\ell(\tilde{\varphi}) \tag{3.6}$$

where

$$\tilde{\varphi}(z) = \frac{(1 + tz^{-1})^{N_1}}{(1 - tz^{-1})^{M_1}} \frac{(1 + tz)^{N_2}}{(1 - tz)^{M_2}}. \tag{3.7}$$

Due to the strong Szegö limit theorem, we obtain a normalization factor Z as

$$Z = \lim_{\ell\to\infty} D_\ell(\tilde{\varphi}) = \frac{(1 + t^2)^{M_1 N_2 + M_2 N_1}}{(1 - t^2)^{M_1 M_2 + N_1 N_2}}. \tag{3.8}$$

Combining these results, we obtain

$$\mathrm{Prob}(L_1 \leqslant \ell) = \frac{1}{Z} D_\ell(\tilde{\varphi}). \tag{3.9}$$

4. Scaling limit

We study the asymptotic behaviour of the probability (3.1). We note that in [18] the property of the scaling limit was conjectured. For our purpose, it is generally useful to rewrite the Toeplitz determinant with the Fredholm determinant. In fact, once we know the Toeplitz determinant, it is possible to rewrite it in terms of the Fredholm determinant [23–25]. Namely we have

$$D_\ell(\tilde{\varphi}) = Z \det(1 - \mathcal{K}_\ell) \tag{4.1}$$

where Z is defined in equation (3.8), and $\mathcal{K}_\ell$ is the matrix defined by

$$\mathcal{K}_\ell(i, j) = \sum_{k=0}^{\infty} (\tilde{\varphi}_-/\tilde{\varphi}_+)_{i+\ell+k+1} (\tilde{\varphi}_+/\tilde{\varphi}_-)_{-j-\ell-k-1}. \tag{4.2}$$

Here a subscript denotes the Fourier component of the function, and we have used the Wiener–Hopf factor of $\tilde{\varphi}$, $\tilde{\varphi} = \tilde{\varphi}_+\tilde{\varphi}_-$,

$$\tilde{\varphi}_+ = \frac{(1+tz)^{N_2}}{(1-tz)^{M_2}} \qquad \tilde{\varphi}_- = \frac{(1+tz^{-1})^{N_1}}{(1-tz^{-1})^{M_1}}.$$

Note that we have set $0 < t < 1$. The probability (3.1) is now written by the Fredholm determinant as

$$\mathrm{Prob}(L_1 \leqslant \ell) = \det(1 - \mathcal{K}_\ell). \tag{4.3}$$

Using a representation (4.3) in terms of the Fredholm determinant, we study an asymptotic behaviour by applying the saddle point method following [10, 26]. We consider a limit $M_a, N_a \to \infty$ for $a = 1, 2$ with fixed values:

$$\frac{M_1}{N_2} = m_1 \qquad \frac{N_1}{N_2} = n_1 \qquad \frac{M_2}{N_2} = m_2.$$

In the Fredholm determinant (4.2), matrix elements are computed as

$$(\tilde{\varphi}_+/\tilde{\varphi}_-)_{-\ell-j-k-1} = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{(1+tz)^{N_2}}{(1-tz)^{M_2}} \cdot \frac{(1-t/z)^{M_1}}{(1+t/z)^{N_1}} z^{j+k+\ell}$$
$$(\tilde{\varphi}_-/\tilde{\varphi}_+)_{\ell+i+k+1} = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{(1-tz)^{M_2}}{(1+tz)^{N_2}} \cdot \frac{(1+t/z)^{N_1}}{(1-t/z)^{M_1}} z^{-i-k-\ell-2}.$$

A path of integration in the former integral is chosen in such a way that it surrounds $z = -t$, and that $z = 1/t$ is outside. On the other hand, a path of the latter integral includes both $z = 0$ and $z = t$ while it excludes $z = -1/t$. We set

$$\ell = cN_2 + sN_2^{\frac{1}{3}} \tag{4.4}$$

where c is to be fixed later. For brevity, we define the function $\sigma(z)$ by

$$\sigma(z) = m_1 \log(t-z) - n_1 \log(t+z) + \log(1+tz) - m_2 \log(1-tz) + (-m_1 + n_1 + c) \log z. \tag{4.5}$$

Then the above integrals are given by

$$(\tilde{\varphi}_+/\tilde{\varphi}_-)_{-\ell-j-k-1} = (-1)^{M_1} \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \mathrm{e}^{N_2\sigma(z)} z^{j+k+sN_2^{1/3}} \equiv (-1)^{M_1} I_1$$
$$(\tilde{\varphi}_-/\tilde{\varphi}_+)_{\ell+i+k+1} = (-1)^{M_1} \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \mathrm{e}^{-N_2\sigma(z)} z^{-sN_2^{1/3}-i-k-2} \equiv (-1)^{M_1} I_2.$$

We scale matrix indices as $(i, j, k) \to \left(N_2^{1/3}x, N_2^{1/3}y, N_2^{1/3}w\right)$, and we consider applying the saddle point method to integrals,

$$I_1 = \int_{\mathcal{C}^+} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \mathrm{e}^{N_2\sigma(z)} z^{N_2^{1/3}(w+y+s)} \qquad I_2 = \int_{\mathcal{C}^-} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \mathrm{e}^{-N_2\sigma(z)} z^{-N_2^{1/3}(w+x+s)-2}$$

in a limit $N_2 \to \infty$. In these integrals, we fix a parameter c in equation (4.4) so that we have a double saddle point, namely as a solution of a set of equations,

$$\frac{\mathrm{d}\sigma(z)}{\mathrm{d}z} = \frac{\mathrm{d}^2\sigma(z)}{\mathrm{d}z^2} = 0$$

i.e.

$$\frac{m_1}{1-z/t}+\frac{1}{1+tz}=c-m_2+1+\frac{m_2}{1-tz}+\frac{n_1}{1+z/t} \tag{4.6a}$$

$$\frac{m_1}{(t-z)^2}-\frac{n_1}{(t+z)^2}=-\frac{c-m_1+n_1}{z^2}+\frac{m_2t^2}{(1-tz)^2}-\frac{t^2}{(1+tz)^2}. \tag{4.6b}$$

This set of equations is rewritten as

$$c=\frac{t}{t-z_0}m_1-\frac{t}{t+z_0}n_1-\frac{tz_0}{1-tz_0}m_2-\frac{tz_0}{1+tz_0} \tag{4.7a}$$

where z_0 satisfies

$$\frac{m_1}{(t-z_0)^2}+\frac{n_1}{(t+z_0)^2}=\frac{1}{(1+tz_0)^2}+\frac{m_2}{(1-tz_0)^2}. \tag{4.7b}$$

We see that equation (4.7b) always has a real solution in $(-1/t,-t)$ as far as $n_1\neq 0$ because lhs–rhs of equation (4.7b) changes from $-\infty$ to ∞ in $z\in(-1/t,-t)$. Generally, real solutions of equation (4.7b) are not only in $(-1/t,-t)$, but to deform paths of integrals adequately we see that $z_0\in(-1/t,-t)$ is a unique candidate of a double saddle point. For example, in the case of $m_1=m_2$ and $n_1=1$, real solutions of equation (4.7b) are only $z=\pm 1$. We can conclude that a double saddle point should be $z_0=-1$ from the discussion below. In the case of $m_1=n_1\geqslant 1$ and $m_2=1$, real solutions of equation (4.7b) are in $(-1/t,-t)$, $(t,1/t)$, $(-\infty,-1/t)$, and $(1/t,\infty)$ (the latter two solutions exist only if $m_1=n_1\geqslant 1/t^2$), and from the discussion to deform contours we see that only $z_0\in(-1/t,-t)$ is possible as a double saddle point. Based on these cases, it may be natural to conclude that we choose $z_0\in(-1/t,-t)$ as a double saddle point.

Hereafter we set a double saddle point z_0 so that $z_0\in(-1/t,-t)$, and we fix a parameter c by equation (4.7a). With $z_0\in(-1/t,-t)$, we find that $c>0$ from a definition (4.7a). With this choice of parameters, the fourth-order equation (4.6a) has a real solution z_0 of multiplicity two, and two other solutions are in $(-t,t)$ and $(1/t,\infty)$. Around z_0, we have the steepest descend path as in figure 4. As z_0 is a double saddle point, paths come into z_0 with angles $\pm\pi/3$ and $\pm 2\pi/3$. Following [10], we denote such paths as $\mathcal{C}^+$ and $\mathcal{C}^-$, respectively. We see that the original paths explained above equation (4.4) can be deformed smoothly to contours $\mathcal{C}^\pm$ avoiding their singularities. Furthermore, we have

$$\begin{aligned}\frac{1}{2}\left.\frac{\mathrm{d}^3\sigma(z)}{\mathrm{d}z^3}\right|_{z=z_0}&=\frac{t}{z_0^2}\left(\frac{t-2z_0}{(t-z_0)^3}m_1+\frac{t+2z_0}{(t+z_0)^3}n_1-\frac{1-2tz_0}{(1-tz_0)^3}m_2-\frac{1+2tz_0}{(1+tz_0)^3}\right)\\&=\frac{-z_0}{1-tz_0}\left(\frac{1-t^2}{(t-z_0)^3}m_1-\frac{1+t^2}{(t+z_0)^3}n_1+\frac{2t}{(1+tz_0)^3}\right)\end{aligned}$$

where in the first equality we have used equation (4.7a) to delete a parameter c, and in the second equality we have erased m_2 using equation (4.7b). Recalling $z_0\in(-1/t,-t)$ and $0<t<1$, we see that

$$\left.\frac{\mathrm{d}^3\sigma(z)}{\mathrm{d}z^3}\right|_{z=z_0}>0 \tag{4.8}$$

which shows that functions $|\mathrm{e}^{\pm N_2\sigma(z_0)}|$ have a maximum value at $z=z_0$ on a contour $\mathcal{C}^\pm$.

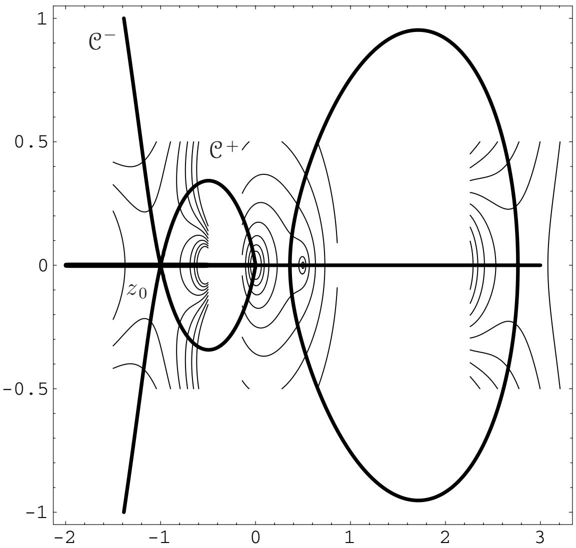


Figure 4. A typical example of the steepest descent path $\mathcal{C}^{\pm}$ is depicted as a bold line. Here we have set $m_1 = n_1 = m_2 = 1$, and $t = 1/2$. A double saddle point is $z_0 = -1$, and other (simple) saddle points are $(25 \pm 3\sqrt{41})/16$. We see that paths $\mathcal{C}^+$ and $\mathcal{C}^-$ come to a double saddle point z_0 at angles $\pm\pi/3$ and $\pm 2\pi/3$ respectively, and that another contour comes into (simple) saddle points with angle $\pm\pi/2$. A thin line denotes a local structure of the real part of the integrand around the saddle points.

With these settings, we have from the integral I_1 that

$$\begin{aligned}
N_2^{1/3} \int_{\mathcal{C}^+} \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, \mathrm{e}^{N_2\sigma(z)} z^{N_2^{1/3}(w+y+s)} \\
&= N_2^{1/3} \mathrm{e}^{N_2\sigma(z_0)} \int_{\mathcal{C}^+} \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, \mathrm{e}^{\frac{N_2}{6}\sigma'''(z_0)(z-z_0)^3} z^{N_2^{1/3}(w+y+s)} \\
&= N_2^{1/3} z_0^{N_2^{1/3}(w+y+s)} \mathrm{e}^{N_2\sigma(z_0)} \int_{\mathcal{C}^+} \frac{\mathrm{d}z}{2\mathrm{i}\pi}\, \mathrm{e}^{\frac{N_2}{6}\sigma'''(z_0)z^3 + N_2^{1/3}\frac{w+y+s}{z_0}z} \\
&= -z_0^{N_2^{1/3}(w+y+s)} \mathrm{e}^{N_2\sigma(z_0)} \frac{z_0}{\sigma} Ai\left(\frac{w+y+s}{\sigma}\right).
\end{aligned}$$

Here $Ai(x)$ is the Airy function,

$$Ai(z) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi}\, \mathrm{e}^{\mathrm{i}(zt+t^3/3)}$$

and we have set a parameter σ as

$$\sigma = -z_0 \left(\frac{1}{2} \left. \frac{\mathrm{d}^3\sigma(z)}{\mathrm{d}z^3} \right|_{z=z_0} \right)^{1/3}. \tag{4.9}$$

We have $\sigma > 0$ from equation (4.8). In the same way, we have from the integral I_2 that

$$N_2^{1/3} \int_{\mathcal{C}^-} \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, \mathrm{e}^{-N_2\sigma(z)} z^{-N_2^{1/3}(w+x+s)-2} = -z_0^{-N_2^{1/3}(w+x+s)}\, \mathrm{e}^{-N_2\sigma(z_0)} \frac{1}{z_0\sigma} Ai\left(\frac{w+x+s}{\sigma}\right).$$

We then see that the kernel of the Fredholm determinant (4.2) is given by the Airy kernel,

$$\frac{1}{\sigma^2} \int_0^{\infty} \mathrm{d}w\, Ai\left(\frac{s+x+w}{\sigma}\right) Ai\left(\frac{s+y+w}{\sigma}\right) = \frac{1}{\sigma} \int_0^{\infty} \mathrm{d}w\, Ai\left(\frac{s+x}{\sigma}+w\right) Ai\left(\frac{s+y}{\sigma}+w\right).$$

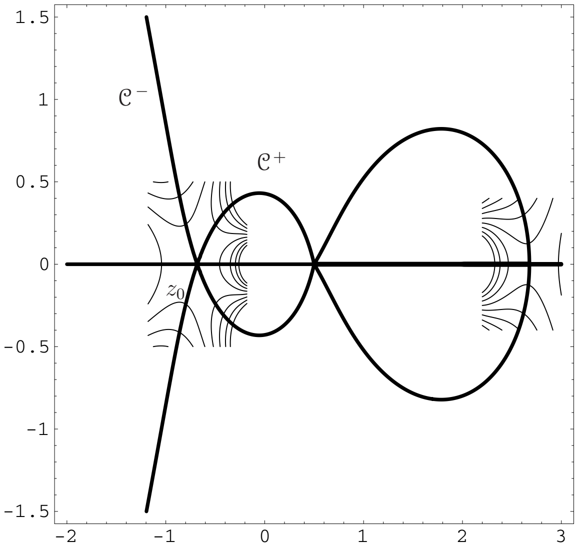


Figure 5. Example of the steepest descent path $\mathcal{C}^{\pm}$ for the case of $n_1 = 0$ is depicted. Here we have set $m_1 = 4, m_2 = 1$, and $t = 1/2$. A double saddle point is $z_0 = -0.682\,54$, and there is a (simple) saddle point at 2.676 84. As in figure 4, paths $\mathcal{C}^+$ and $\mathcal{C}^-$ come to z_0 at angles $\pm\pi/3$ and $\pm 2\pi/3$, respectively. Another contour comes into a (simple) saddle point with angle $\pm\pi/2$, and it ends at t. A thin line denotes a local structure of the real part of the integrand around saddle points.

As a result, we obtain

$$\lim_{N_2\to\infty} \text{Prob}\left(\frac{L_1 - cN_2}{\sigma N_2^{1/3}} \leqslant s\right) = F_2(s). \tag{4.10}$$

Here $F_2(s)$ is the Tracy–Widom distribution [12] for the scaling limit of the largest eigenvalue of the Gaussian unitary ensemble, and is defined by

$$F_2(s) = \det(1 - \mathcal{K}_{\text{Airy}}) \tag{4.11}$$

$$= \exp\left(-\int_s^\infty \mathrm{d}x\,(x-s)q(x)^2\right). \tag{4.12}$$

The second equality is from [12], and $q(x)$ is a solution of the Painlevé II equation,

$$q'' = sq + 2q^3 \tag{4.13}$$

with $q(s) \to Ai(s)$ in $s \to \infty$.

A proof of convergence would be performed along the line of [10, 26].

To close this section, we comment on the case of $n_1 = 0$. In this case, we further suppose that $m_1/t^2 > 1 + m_2$. With this assumption, we see that there exists a real solution of equation (4.7*b*) in $(-1/t, 0)$. By setting this real solution to be $z_0 \in (-1/t, 0)$, we can prove from equations (4.7*a*) and (4.9) that $c > 0$ and $\sigma > 0$. Note that, with this setting of a parameter c, equation (4.6*a*) has a real solution z_0 of multiplicity two, and another solution is in $(1/t, \infty)$. See figure 5 for an example of the steepest descent path. As a result, we obtain the Tracy–Widom distribution (4.10) as a scaling limit.

5. Some special cases

5.1. Meixner ensemble

We consider a case

$$M_1 = M_2 = 0 \qquad \text{i.e.} \quad m_1 = m_2 = 0.$$

From the point of view of the random walkers, the vicious walkers move obeying only the super time evolution rule. In this case, the Toeplitz determinant (3.6) reduces to

$$D_\ell\left(\left(1+\frac{t}{z}\right)^{N_1}(1+tz)^{N_2}\right) = \sum_n \sum_{\substack{\ell(\lambda)\leqslant \ell\\ \lambda\vdash n}} d_{\lambda'}(N_1)d_{\lambda'}(N_2)t^{2n}$$
$$= \sum_n \sum_{\substack{\mu_1\leqslant \ell\\ \mu\vdash n}} d_\mu(N_1)d_\mu(N_2)t^{2n}$$

where $d_\lambda(N)$ denotes the number of (usual) SSYT, and we have $d_\lambda(N) = d_\lambda(N,0) = d_{\lambda'}(0,N)$ from equation (1.5). The right-hand side (rhs) appeared in [7], and it gives an example of the discrete orthogonal polynomial ensemble as follows. Using the hook formula [22],

$$d_\mu(M) = \prod_{1\leqslant i<j\leqslant M} \frac{\mu_i - \mu_j + j - i}{j-i} \tag{5.1}$$

the rhs gives

$$\sum_n \sum_{\substack{\mu_1\leqslant \ell\\ \mu\vdash n}} \left(\prod_{1\leqslant i<j\leqslant N_2}\left[\frac{\mu_i-\mu_j+j-i}{j-i}\right]^2 \cdot \left(\prod_{i=1}^{N_2}\left[\prod_{j=N_2-1}^{N_1}\frac{\mu_i+j-i}{j-i}\right]t^{2\mu_i}\right)\right)$$

where we have assumed $N_1 \geqslant N_2$. Introducing

$$h_j = \mu_j + N_2 - j \tag{5.2}$$

we obtain

$$\text{Prob}(L_1 \leqslant \ell) = (1-t^2)^{N_1N_2}t^{-N_2(N_2-1)}\left[\prod_{j=0}^{N_2-1}\frac{(N_1-N_2)!}{j!(N_1-N_2+j)!}\right]$$
$$\times \sum_{\substack{h\in\mathbb{N}^{N_2}\\ \max\{h_i\}\leqslant \ell+N_2-1}}\left[\prod_{1\leqslant i<j\leqslant N_2}(h_i-h_j)^2\right]\prod_{i=1}^{N_2}\binom{N_1-N_2+h_i}{h_i}t^{2h_i} \tag{5.3}$$

which is called the Meixner ensemble.

In fact, using the Borodin–Okounkov identity (4.2), the kernel of the Fredholm determinant can be written in terms of the Meixner polynomial

$$(i-j)\mathcal{K}(i,j) = t^{4N_2+i+j}(1-t^2)^{N_1-N_2-1}\binom{N_1+j}{N_2+j}\cdot\binom{N_1}{N_2}\cdot(-N_2)$$
$$\times(M_{N_2}(i+N_2;N_1-N_2+1,t^2)\cdot M_{N_2-1}(j+N_2;N_1-N_2+1,t^2)$$
$$-M_{N_2-1}(i+N_2;N_1-N_2+1,t^2)\cdot M_{N_2}(j+N_2;N_1-N_2+1,t^2)) \tag{5.4}$$

which has a well-known form of the correlation functions of the random matrix (see, for example, [27]). Note that the Meixner polynomial is defined by

$$M_n(x;b,c) = {}_2F_1\left(\begin{matrix}-n,-x\\ b\end{matrix};1-\frac{1}{c}\right).$$

A computation of the scaling limit can be done by the method in section 4. A double saddle point, $z_0 \in (-1/t, -t)$, is explicitly solved as

$$z_0 = -\frac{t + \sqrt{n_1}}{1 + t\sqrt{n_1}}$$

and we obtain the Tracy–Widom distribution (4.10) with parameters c and σ defined by

$$c = \frac{t(2\sqrt{n_1} + (n_1 + 1)t)}{1 - t^2} \tag{5.5}$$

$$\sigma = \frac{t^{1/3}(\sqrt{n_1} + t)^{2/3}(1 + t\sqrt{n_1})^{2/3}}{n_1^{1/6}(1 - t^2)}. \tag{5.6}$$

This result was derived by using the asymptotics of the Meixner polynomial in [7] (see also [28]).

5.2. Krawtchouk ensemble

We set

$$N_1 = M_2 = 0 \qquad \text{i.e.} \quad n_1 = m_2 = 0 \tag{5.7}$$

The vicious walkers obey a rule of normal time evolution in moving right, while they obey a rule of super time evolution in moving left. In this case, equation (3.6) is read as

$$D_\ell\left(\frac{(1 + tz)^{N_2}}{\left(1 - \frac{t}{z}\right)^{M_1}}\right) = \sum_n \sum_{\substack{\ell(\lambda)\leqslant\ell \\ \lambda\vdash n}} d_\lambda(M_1) d_{\lambda'}(N_2) t^{2n}.$$

This becomes the Krawtchouk ensemble [29] as follows (this type of the Toeplitz determinant was also studied in [10]). When we substitute the hook formula (5.1) into the above expression, we see that the rhs reduces to

$$\left[\prod_{j=0}^{N_2-1} \frac{(M_1 + j)!}{j!}\right] \sum_n \sum_{\substack{\mu_1\leqslant\ell \\ \mu\vdash n}} \left[\prod_{1\leqslant i<j\leqslant N_2} (\mu_i - \mu_j + j - i)^2\right]$$
$$\times \prod_{j=1}^{N_2} \frac{t^{2\mu_j}}{(\mu_j + N_2 - j)!(M_1 + j - 1 - \mu_j)!}.$$

By use of

$$h_j = \mu_j + N_2 - j$$

this gives

$$\text{Prob}(L_1 \leqslant \ell) = (1 + t^2)^{-M_1 N_2} t^{-N_2(N_2-1)} \left[\prod_{j=0}^{N_2-1} \frac{(M_1 + j)!}{j!(N_2 + M_1 - 1)!}\right]$$
$$\times \sum_{\substack{h\in\mathbb{N}^{N_2} \\ \max\{h_i\}\leqslant\ell+N_2-1}} \left[\prod_{1\leqslant i<j\leqslant N_2} (h_i - h_j)^2\right] \prod_{i=1}^{N_2} \binom{M_1 + N_2 - 1}{h_i} t^{2h_i} \tag{5.8}$$

which is the Krawtchouk ensemble.

The kernel of the Fredholm determinant is computed explicitly from equation (4.2), and it is given in terms of the Krawtchouk polynomial as a form of the correlation functions:

$$
\begin{aligned}
(i-j)\mathcal{K}(i,j) = {} & -\frac{t^{i+j+4N_2}}{(1+t^2)^{M_1+N_2}}(M_1+N_2-1)\binom{M_1+N_2-1}{M_1-1}\cdot\binom{M_1+N_2-1}{N_2+j} \\
& \times\left(K_{N_2}\left(i+N_2;\frac{t^2}{1+t^2},M_1+N_2-1\right)\right. \\
& \times K_{N_2-1}\left(j+N_2;\frac{t^2}{1+t^2},M_1+N_2-1\right) \\
& - K_{N_2-1}\left(i+N_2;\frac{t^2}{1+t^2},M_1+N_2-1\right) \\
& \left.\times\, K_{N_2}\left(j+N_2;\frac{t^2}{1+t^2},M_1+N_2-1\right)\right).
\end{aligned}
\tag{5.9}
$$

Here the Krawtchouk polynomial is defined by

$$
K_n(x;p,N) = {}_2F_1\left(\begin{matrix}-n,-x\\-N\end{matrix};\frac{1}{p}\right).
$$

We note that we have

$$
K_n(x;p,N) = M_n\left(x;-N,\frac{p}{p-1}\right).
$$

The scaling limit is also computed by the saddle point method [10]. In this case we suppose $m_1 > t^2$, and we have a double saddle point $z_0 \in (-1/t, 0)$ as

$$
z_0 = \frac{-\sqrt{m_1}+t}{1+t\sqrt{m_1}}.
$$

We obtain the Tracy–Widom distribution (4.10) where parameters c and σ are defined from equations (4.7a) and (4.9) as

$$
c = \frac{t(2\sqrt{m_1}+(m_1-1)t)}{1+t^2}
\tag{5.10}
$$

$$
\sigma = \frac{t^{1/3}(\sqrt{m_1}-t)^{2/3}(1+t\sqrt{m_1})^{2/3}}{m_1^{1/6}(1+t^2)}.
\tag{5.11}
$$

We see that this result coincides with that of [29] derived by use of asymptotics of the Krawtchouk polynomial.

5.3. Symmetric case

We consider a case

$$
M_1 = M_2 \qquad N_1 = N_2 \qquad \text{i.e.} \quad m_1 = m_2 = m \quad n_1 = 1
\tag{5.12}
$$

namely in right and left movements we have an equal number of normal and super time evolutions. Unfortunately, we are not sure whether this model is related to the discrete orthogonal ensemble, but the parameters of the scaling function can be simply solved as follows.

In a scaling limit $N_2 \to \infty$, we obtain the Tracy–Widom distribution (4.10) by applying the saddle point method. In this case, a double saddle point is $z_0 = -1$, and parameters in equation (4.10) are computed from equations (4.7a) and (4.9) as

$$c = \frac{2t(1+m+(1-m)t)}{1-t^2} \tag{5.13}$$

$$\sigma = \frac{t^{1/3}(m(1-t)^4+(1+t)^4)^{1/3}}{1-t^2}. \tag{5.14}$$

6. Conclusion and discussion

We have introduced a generalization of the vicious walker model in [1]. We find that there exists a bijection map between the path configuration of vicious walkers and the hook Young diagram as in the case of the original vicious walkers. We have exactly computed a probability that the number of right movements of the first walker is less than ℓ, and we have given a formula in terms of the Toeplitz determinant. We have further studied a scaling limit of the probability based on the Borodin–Okounkov identity which relates the Toeplitz determinant with the Fredholm determinant, and we have obtained the Tracy–Widom distribution for the largest eigenvalue of the Gaussian unitary random matrix. Other models which belong to the orthogonal or the symplectic universality classes are for future studies.

In the case of the vicious walker model, the crucial point is that there exists the bijection map from the path configuration to a pair of the semi-standard (hook) Young tableaux. As has been well studied [6], a pair of SSYT and the standard tableaux are related to the problem of the random word. We can define the model of the random word, which is related to the hook Young diagram as follows [30]. We consider a random word by choosing from a set $\mathbf{B}_+ \sqcup \mathbf{B}_-$ with $\mathbf{B}_+ = \{1, \ldots, M\}$ and $\mathbf{B}_- = \{M+1, \ldots, M+N\}$. When a word of length n is given, we have a generalization of the Robinson–Schensted–Knuth (RSK) correspondence [31, 32] (see also [33] for invariance under ordering of symbols); we have a bijection between a word of length n and pairs (P, Q) of tableaux of the same shape $\lambda \vdash n$ (P is SSYT from $\mathbf{B}$, and the recording tableaux Q is the standard Young tableaux). The rule to construct pairs of tableaux is essentially the same with the original RSK correspondence (see, for example, [21, 22]), and the difference is only that negative symbols can bump together while positive symbols cannot. Then for a random word with length n, the probability that the length of the longest decreasing (strictly decreasing for positive symbols while weakly decreasing for negative symbols) subsequence is less than or equal to ℓ is then given by

$$\sum_{\substack{\ell(\lambda)\leqslant\ell \\ \lambda\vdash n}} d_\lambda(M, N) f^\lambda \tag{6.1}$$

where f^λ is the number of standard Young tableaux.

This can be rewritten in terms of the Toeplitz determinant based on equation (3.3). We use the exponential specialization [22],

$$\mathrm{ex}(p_n) = t\delta_{1,n} \tag{6.2}$$

where the power sum symmetric function p_n is given by

$$p_n(x, y) = \sum_i x_i^n + (-1)^{n-1} \sum_j y_j^n$$

Acting on the hook Schur function, we have

$$\mathrm{ex}(S_\lambda(x, y)) = f^\lambda \frac{t^n}{n!}$$

for $\lambda \vdash n$. By applying the exponential specialization to (x, y) and the principal specialization $\mathrm{ps}_{a=b=t;q=1}$ to (z, w) in equation (3.3), we obtain

$$\sum_{n} \left(\sum_{\substack{\ell(\lambda)\leqslant \ell \\ \lambda \vdash n}} d_\lambda(M, N) f^\lambda \right) \frac{t^n}{n!} = D_\ell(\Phi) \tag{6.3}$$

where

$$\Phi(z) = \mathrm{e}^{t/z} \frac{(1+z)^M}{(1-z)^N}. \tag{6.4}$$

As a consequence, the Poisson generating function of the probability (6.1) is given by the Toeplitz determinant of function Φ. As seen from the fact that the kernel (6.4) can be given from equation (3.7) as an appropriate limit, the scaling limit of equation. (6.3) reduces to the Tracy–Widom distribution, as was shown in [16] for a case of $N = 0$. Details will be discussed elsewhere.

It was shown in [6] that the generating function (6.3) with $N = 0$ has an integral representation in terms of solutions of the Painlevé V equation. It remains for future studies to clarify a relationship between the Toeplitz determinant (6.3) in a case of $N \neq 0$ and the Painlevé equations, especially the integral solutions of the Painlevé equation given in [34].

Note added: After submitting this paper, Tracy and Widom [35] appeared on the Internet. Therein, a limit theorem of the 'shifted Schur measure' was studied, where the probability is defined in terms of the Schur Q-functions [20]. To apply a method of [10], they obtained the Fredholm determinant after a finite perturbation of a product of Hankel operator, but their main result on a scaling limit exactly coincides with our results (4.10) with $M_1 = N_1$ and $M_2 = N_2$ (subsequently we see that their result for $\tau = 1$ coincides with our above results (5.13)–(5.14) with $m = 1$). This coincidence may originate from a property of the Schur Q-function. The Schur Q-function is defined by filling 'marked' and 'unmarked' positive integers to the shifted Young diagram; a rule of filling these numbers is much the same as a rule for the semi-standard hook Young tableaux explained in the introduction, once we identify unmarked (marked) numbers with positive (negative) symbols. It will be interesting to investigate this connection in detail.

References

[1] Fisher M E 1984 *J. Stat. Phys.* **34** 667
[2] Guttmann A J, Owczarek A L and Viennot X G 1998 *J. Phys. A: Math. Gen.* **31** 8123
[3] Forrester P J 2001 *J. Phys. A: Math. Gen.* **34** L417
[4] Baik J 2000 *Commun. Pure Appl. Math.* **53** 1385
[5] Baik J, Deift P and Johansson K 1999 *J. Am. Math. Soc.* **12** 1119
[6] Tracy C A and Widom H 2001 *Probab. Theory Relat. Fields* **119** 381
[7] Johansson K 2000 *Commun. Math. Phys.* **209** 437
[8] Seppäläinen T. 2001 *Ann. Probab.* **29** 176
[9] Prähofer M and Spohn H 2000 *Physica* A **279** 342
[10] Gravner J, Tracy C A and Widom H 2001 *J. Stat. Phys.* **102** 1085
[11] Baryshnikov Y 2001 *Probab. Theory Relat. Fields* **119** 256
[12] Tracy C A and Widom H 1994 *Commun. Math. Phys.* **159** 151
[13] Aldous D and Diaconis P 1999 *Bull. Am. Math. Soc.* **36** 413
[14] Deift P 2000 *Not. Am. Math. Soc.* **47** 631
[15] Baik J and Rains E M 2001 *Random Matrix Models and their Applications* ed P Bleher and A Its *(Mathematical Sciences Research Institute Publications vol 40)* (Cambridge: Cambridge University Press) pp 1–19
[16] Johansson K 2001 *European Congress of Mathematics* vol 1 ed C Casacuberta, R Miro-Roig, J Verdera and S Xanbo-Descamps *(Prog. Math. vol 201)* (Basle: Birkhauser) pp 445–56
[17] Baik J and Rains E M 2001 *Duke Math. J.* **109** 1
[18] Rains E M 2000 *Preprint* math.CO/0004082
[19] Berele A and Regev A 1987 *Adv. Math.* **64** 118
[20] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* 2nd edn (Oxford: Oxford University Press)

[21] Fulton W 1997 *Young Tableaux (London Mathematical Society Student Texts vol 35)* (Cambridge: Cambridge University Press)
[22] Stanley R P 1999 *Enumerative Combinatorics* vol 2 (Cambridge: Cambridge University Press)
[23] Borodin A and Okounkov A 2000 *Integr. Equ. Oper. Theory* **37** 386
[24] Basor E L and Widom H 2000 *Integr. Equ. Oper. Theory* **37** 397
[25] Böttcher A. 2001 *Integr. Equ. Oper. Theory* **41** 123
[26] Gravner J, Tracy C A and Widom H 2002 *Ann. Probab.* **30** 1340
[27] Mehta M L 1991 *Random Matrices* (New York: Academic)
[28] Baik J, Deift F, McLaughlin K, Miller P and Zhou X 2001 *Preprint* math.PR/0112162
[29] Johansson K 2001 *Ann. Math.* **153** 259
[30] Fulman J 2001 *Preprint* math.CO/0104003
[31] Berele A and Remmel J 1985 *J. Pure Appl. Algebra* **35** 225
[32] Kerov S and Vershik A 1986 *SIAM J. Algebr. Discrete Methods* **7** 116
[33] Regev A and Seeman T 2002 *Adv. Appl. Math.* **28** 59
[34] Forrester P J and Witte N S 2002 *Preprint* math-ph/0204008
[35] Tracy C A and Widom H 2002 *Preprint* math.PR/0210255